GRAVITATIONAL TURBULENT INSTABILITY OF AdS$_5$

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We consider the problem of stability of anti-de Sitter spacetime in five dimensions under small purely gravitational perturbations satisfying the cohomogeneity-two biaxial Bianchi IX Ansatz. In analogy to spherically symmetric scalar perturbations, we observe numerically a black hole formation on the time-scale $\mathcal{O}(\varepsilon^{-2})$, where ε is the size of the perturbation.

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1. Introduction

Over the past nearly two decades, asymptotically anti-de Sitter (aAdS) spacetimes have received a great deal of attention, primarily due to the AdS/CFT correspondence which is the conjectured duality between aAdS spacetimes and conformal field theories. The distinctive feature of aAdS spacetimes, on which the very concept of duality rests, is a timelike conformal boundary at spatial and null infinity, where it is necessary to specify boundary conditions in order to define the deterministic evolution. For energy conserving boundary conditions, the conformal boundary acts as a mirror at which massless waves propagating outwards bounce off and return to the bulk. Therefore, the key mechanism stabilizing the evolution of asymptotically flat spacetimes — dispersion of energy by radiation — is absent in aAdS spacetimes. For this reason, the problem of nonlinear stability of the pure AdS spacetime (which is the ground state among aAdS spacetimes) is particularly challenging.

A few years ago, we considered this problem in a toy model of the spherically symmetric massless scalar field minimally coupled to gravity with a negative cosmological constant in four and higher dimensions and gave evidence for the instability of the AdS spacetime [1, 2]. More precisely, we showed numerically that there is a large class of arbitrarily small perturbations of AdS that evolve into a black hole on the time-scale $\mathcal{O}(\varepsilon^{-2})$, where ε is the size of the perturbation. On the basis of nonlinear perturbation analysis, we conjectured that this instability is due to a resonant transfer of energy from low to high frequencies or, equivalently, from coarse to fine spatial scales, until eventually an apparent horizon forms.

Further studies of this and similar models confirmed and extended our findings, and provided important new insights concerning the coexistence of unstable (turbulent) and stable (quasiperiodic) regimes of evolution (see [3] for a brief review and references).

The major downside of all the reported numerical simulations of AdS instability was the restriction to spherical symmetry, so that no gravitational degrees of freedom were excited. Nonetheless, the results of nonlinear perturbation analysis of the vacuum Einstein equation without symmetry assumptions [4] seem to indicate that the instability mechanism is present for gravitational perturbations as well. To verify this expectation, in this paper, we consider the vacuum Einstein equations with negative cosmological constant in five dimensions within the cohomogeneity-two biaxial Bianchi IX Ansatz, introduced originally in the context of critical collapse for asymptotically flat spacetimes [5]. This Ansatz provides a simple 1+1 dimensional setting for analyzing stability of AdS_5 (which, incidentally, happens to be dimensionwise the most interesting case from the AdS/CFT viewpoint) under purely gravitational perturbations. As expected, we observe a similar instability phenomenon as in [1, 2]. In the remaining two sections, we describe the model and present numerical results.

2. Setup

We consider the vacuum Einstein equations with the negative cosmological constant in five dimensions

$$R_{\alpha\beta} = -\frac{4}{\ell^2}\, g_{\alpha\beta}\,. \tag{1}$$

Following [5], we assume the cohomogeneity-two biaxial Bianchi IX Ansatz

$$g = \frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}\mathrm{d}t^2 + A^{-1}\mathrm{d}x^2 + \frac{1}{4}\sin^2 x\left(e^{-2B}\left(\sigma_1^2+\sigma_2^2\right) + e^{4B}\sigma_3^2\right)\right), \tag{2}$$

where $(t,x)\in\mathbb{R}\times[0,\pi/2)$ and A, δ, B are functions of (t,x). The angular part of this metric is the $\mathrm{SU}(2)\times\mathrm{U}(1)$-invariant homogeneous metric on the squashed 3-sphere. Here, σ_k are left-invariant one-forms on SU(2) which in terms of Euler angles $(0\le\vartheta\le\pi,\ 0\le\varphi,\psi\le 2\pi)$ take the form of

$$\sigma_1 + i\sigma_2 = e^{i\psi}(\cos\vartheta\mathrm{d}\varphi + i\mathrm{d}\vartheta)\,, \qquad \sigma_3 = \mathrm{d}\psi - \sin\vartheta\mathrm{d}\varphi\,. \tag{3}$$

Note that if $B=0$, the angular metric becomes the round metric on S^3 and the symmetry is enhanced to SO(4).

For Ansatz (2), the Einstein equations (1) reduce to the following 1+1 dimensional system (hereafter, primes and overdots denote derivatives with respect to x and t, respectively):

$$\dot{B} = Ae^{-\delta}P\,, \qquad \dot{P} = \frac{1}{\tan^3 x}\left(\tan^3 x\, Ae^{-\delta}Q\right)' - \frac{4e^{-\delta}}{3\sin^2 x}\left(e^{-2B} - e^{-8B}\right)\,, \tag{4}$$

$$A' = 4\tan x\,(1-A) - 2\sin x\cos x\, A\left(Q^2+P^2\right) + \frac{2(4e^{-2B} - e^{-8B} - 3A)}{3\tan x}\,, \tag{5}$$

$$\delta' = -2\sin x\cos x\left(Q^2+P^2\right)\,, \tag{6}$$

$$\dot{A} = -4\sin x\cos x\, A^2 e^{-\delta} QP\,, \tag{7}$$

where we have introduced the auxiliary variables $Q = B'$ and $P = A^{-1}e^{\delta}\dot{B}$. The field B is the only dynamical degree of freedom which plays a role similar to the spherical scalar field[1]. It is convenient to define the mass function

$$m(t,x) = \frac{\sin^2 x}{\cos^4 x}\left(1 - A(t,x)\right). \tag{8}$$

From the Hamiltonian constraint (5), it follows that

$$m'(t,x) = 2\left[A\left(Q^2+P^2\right) + \frac{1}{3\sin^2 x}\left(3 + e^{-8B} - 4e^{-2B}\right)\right]\tan^3 x \geq 0\,. \tag{9}$$

We want to solve system (4)–(6) for small smooth initial data with finite total mass $M = \lim_{x\to\pi/2} m(t,x)$. Smoothness at $x=0$ implies that

$$B(t,x) = b_0(t)\,x^2 + \mathcal{O}\left(x^4\right)\,, \qquad \delta(t,x) = \mathcal{O}\left(x^4\right)\,, \qquad A(t,x) = 1 + \mathcal{O}\left(x^4\right)\,, \tag{10}$$

where we used normalization $\delta(t,0) = 0$ to ensure that t is the proper time at the origin. The power series (10) are uniquely determined by the free function $b_0(t)$. Smoothness at $x = \pi/2$ and finiteness of the total mass M imply that (using $\rho = x - \pi/2$)

$$\begin{aligned} B(t,x) &= b_\infty(t)\,\rho^4 + \mathcal{O}\left(\rho^6\right)\,, \qquad \delta(t,x) = \delta_\infty(t) + \mathcal{O}\left(\rho^8\right)\,,\\ A(t,x) &= 1 - M\rho^4 + \mathcal{O}\left(\rho^6\right)\,, \end{aligned} \tag{11}$$

[1] If $B = 0$, the only solution is the Schwarzschild–AdS family, in agreement with the Birkhoff theorem.

where the free functions $b_\infty(t)$, $\delta_\infty(t)$, and mass M uniquely determine the power series. It follows from (11) that the asymptotic behavior of fields at infinity is completely fixed by the assumptions of smoothness and finiteness of total mass, hence there is no freedom of imposing the boundary data.

The pure AdS spacetime corresponds to $B=0$, $A=1$, $\delta=0$. Linearizing around this solution, we obtain

$$\ddot{B} + LB = 0\,, \qquad L = -\frac{1}{\tan^3 x}\,\partial_x \left(\tan^3 x\, \partial_x\right) + \frac{8}{\sin^2 x}\,. \tag{12}$$

This equation is the $\ell = 2$ gravitational tensor case of the master equation describing the evolution of linearized perturbations of AdS spacetime, analyzed in detail by Ishibashi and Wald [6]. The Sturm–Liouville operator L is essentially self-adjoint with respect to the inner product $(f,g) = \int_0^{\pi/2} f(x)g(x)\tan^3 x\,\mathrm{d}x$. The eigenvalues and associated orthonormal eigenfunctions of L are ($k=0,1,\dots$)

$$\begin{aligned} \omega_k^2 &= (6+2k)^2\,, \\ e_k(x) &= 2\sqrt{\frac{(k+3)(k+4)(k+5)}{(k+1)(k+2)}}\,\sin^2 x\cos^4 x\, P_k^{(3,2)}(\cos 2x)\,, \end{aligned} \tag{13}$$

where $P_k^{(a,b)}(x)$ is a Jacobi polynomial of the order of k.

The eigenfunctions $e_k(x)$ fulfill the regularity conditions (10) and (11) hence any smooth solution can be expressed as

$$B(t,x) = \sum_{k\geq 0} b_k(t) e_k(x)\,. \tag{14}$$

To quantify the transfer of energy between the modes, we introduce the linearized energy

$$E = \int\limits_0^{\pi/2} \left(\dot{B}^2 + B'^2 + \frac{8}{\sin^2 x} B^2\right) \tan^3 x\,\mathrm{d}x = \sum_{k\geq 0} E_k\,, \tag{15}$$

where $E_k = \dot{b}_k^2 + \omega_k^2 b_k^2$ is the energy of the k^{th} mode.

3. Numerical results

We solve system (4)–(6) numerically for small smooth initial data and the boundary conditions (10) and (11). We use the standard method of lines with a fourth-order Runge–Kutta time integration and fourth-order spatial finite differences. The Kreiss–Oliger dissipation is added to eliminate high-frequency instabilities. The scheme is fully constrained, that is the metric functions A and δ are updated at each time step by solving the Hamiltonian constraint (5) and the slicing condition (6). The momentum constraint (7) is only monitored to verify the accuracy of computations. For the typical initial data, the energy is rapidly transferred to small spatial scales. To resolve these scales, we refine the entire spatial grid when a global spatial error exceeds some prescribed tolerance level. We usually start on a grid with $2^{14}+1$ points and allow for four levels of refinement. To feel confident that the spatial scales are properly resolved, we validated our computations by convergence tests.

The results presented below were generated from the Gaussian initial data of the form of

$$B(0,x)=0\,,\qquad P(0,x)=\varepsilon\left(\frac{2}{\pi}\right)^3\frac{512}{\sqrt{3}}x^2\exp\left(-4\tan^2 x/\left(\pi^2\sigma^2\right)\right) \tag{16}$$

with width $\sigma = 1/16$ and varying small amplitudes ε. A good indicator of instability is the Kretschmann scalar at the origin

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(t,0)=40+864\,Q'(t,0)^2\,. \tag{17}$$

For initially small narrow wave packets such as (16), this quantity oscillates with a period nearly equal to π, while the amplitude of oscillations grows exponentially. As shown in Fig. 1, the one-period maxima of the quantity $\varepsilon^{-2}Q'(\varepsilon^2 t,0)$ are almost independent of ε.

At the end of evolutions shown in Fig. 1, we observe the formation of an apparent horizon which is signaled by $A(t,r)$ dropping below a certain small threshold (we take this threshold to be 2^{-7} on the $N=2^{14}$ grid and then divide it by two each time we increase the grid resloution by a factor two). Since the computational costs of numerical simulations rapidly increase with decreasing ε, we have not been able to determine the outcome of evolution of smaller perturbations. Nonetheless, extrapolating the observed scaling behavior, we conjecture that AdS_5 is unstable against black hole formation for a large class of arbitrarily small purely gravitational perturbations.

On a heuristic level, the mechanism of instability is the same as for scalar perturbations, namely the turbulent cascade of energy from low to high modes that is eventually cut-off by the formation of a black hole[2]. To substantiate this claim, let us see how the energy of the perturbation gets distributed over the modes in the course of evolution. To this end, in Fig. 2, we depict the linear energy spectrum, as defined in (15). The range

[2] We stress that this instability is not active for some perturbations. In particular, there exist initial data for which the solutions are exactly time-periodic [7].

of modes participating in the evolution is seen to increase very rapidly. Just before collapse, the spectrum exhibits the power-law scaling $E \sim k^{-\alpha}$, where the exponent $\alpha \approx 5/3$ appears to be universal, *i.e.*, independent of initial data (for comparison, in the Einstein-scalar AdS collapse in five dimensions $\alpha \approx 2$ [8]). This power-law spectrum reflects the loss od smoothness of the collapsing solution.

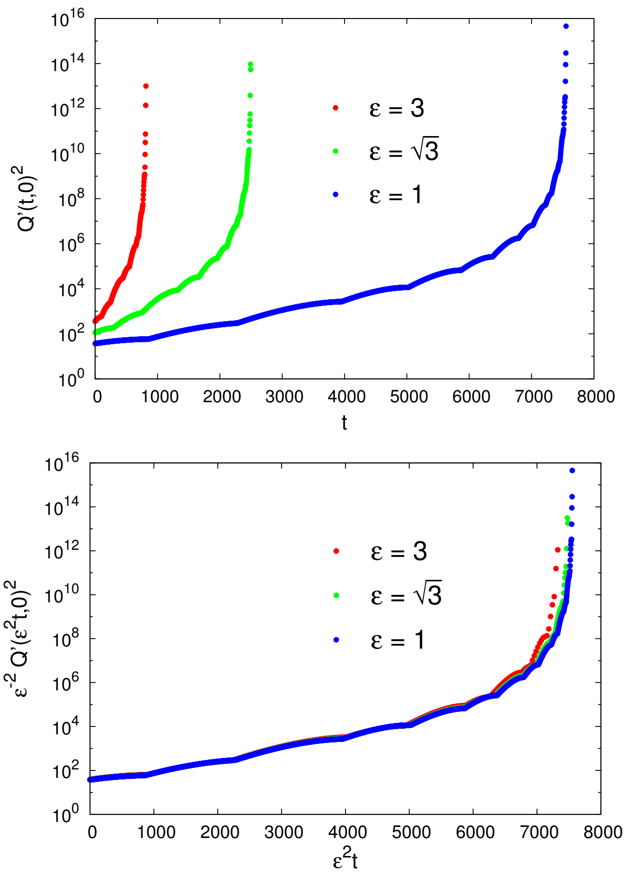

Fig. 1. Top: One-period maxima of $Q'(t,0)^2$ for the initial data (16) with three relatively small amplitudes (note, however, that for these data $Q'(0,0)^2$ is much bigger than the unperturbed Kretschmann scalar at the origin). Bottom: After rescaling $\varepsilon^{-2}Q'(\varepsilon^2 t)^2$, the curves from the top panel nearly coincide.

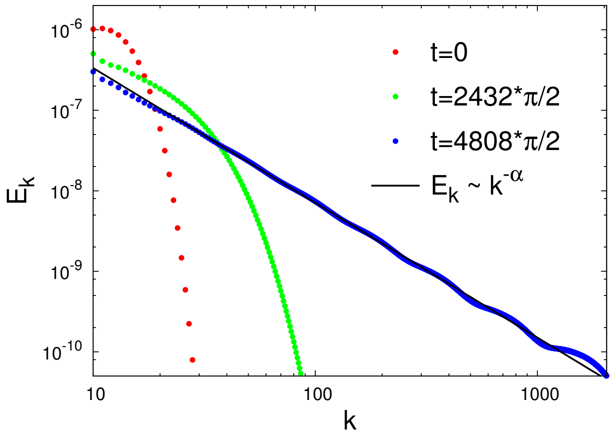

Fig. 2. Log–log plot of the linear energy spectra at three instants of time for the initial data (16) with $\varepsilon = 0.3$. The fit at $t \approx 4808\pi/2$ yields a power-law spectrum $E_k \sim k^{-\alpha}$ with $\alpha \approx 5/3$.

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